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Metastability for the Contact Process

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We prove that the Harris contact process shows metastable behavior for any supercritical value of the parameter, even when some macroscopic observables are observed.

KEY WORDS: Metastability; contact process.

1. INTRODUCTION

The problem of explaning theoretically the phenomenon of metastability as shown by undercooled gases, ferromagnets along the hysteresis loop, and many other systems has been attacked in many different ways during the last years.[5,6] In the present paper we adopt the theory proposed recently in Ref. 1 (see also Ref. 10).

In this theory a system is in a metastable situation if (a) it stays out of its equilibrium situation during a memoryless random time (exponential random time), and (b) during this time in which the system is out of equilibrium it stabilizes in the sense that an observer measuring temporal means of some observable quantity will record values which are close to the expectation of this observable in some fixed probability distribution on the configurations of the system. We call this property *thermalization.*

In the present work we extend the theorems proved in Ref. 1 for the Harris contact process (this process was invented in Ref. 9; for reviews see Refs. 3, 4, and 8) concerning its metastable behavior.

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Informally this process can be described as follows: particles are distributed in $\mathbb{Z}$ in such a way that each site is either empty or occupied by at most one particle. The stochastic time evolution of the system is Markovian and has the following features: particles disappear at rate 1 and new particles appear at each empty site $x \in \mathbb{Z}$ at rate $\lambda \cdot$ (number of particles at the sites $x-1$ and $x+1$). $\lambda \in \mathbb{R}_+$ is a fixed parameter. Precise definitions are left to the next section.

The configurations of the process can be identified with the subsets of $\mathbb{Z}$ in the usual way: the set of occupied sites defines the configuration. It is clear that $\varnothing$ (all sites empty) is a trap for the process, so $\delta_{\varnothing}$ is an invariant measure for every $\lambda \in \mathbb{R}_+$. But one of the fundamental results about that model is the existence of a critical value $\lambda_* \in (0, \infty)$ such that for $\lambda > \lambda_*$ there exists another extremal invariant probability measure which we denote by μ. For $\lambda < \lambda_*$ $\delta_{\varnothing}$ is the only invariant probability measure.

Consider also the finite analog of this model in which particles are distributed in $\{-N, -N+1, \ldots, N-1, N\}$, $N \in \mathbb{N}$, and the stochastic evolution is as before except that a new particle appears at the site $-N$ (resp. N) when it is empty at rate $\lambda \cdot$ (number of particles at site $-N+1$) [resp. $\lambda \cdot$ (number of particles at site $N-1$)]. For any $\lambda \in \mathbb{R}_+$ and any initial configuration the process will eventually reach $\varnothing$ and stay there. $\delta_{\varnothing}$ is the only stable equilibrium.

In Ref. 1 it was shown that starting with all sites in $\{-N, -N+1, \ldots, N-1, N\}$ occupied the finite model presents a metastable behavior before it reaches the equilibrium $\delta_{\varnothing}$, if N is large. But there one used the technical hypothesis that $\lambda > \boldsymbol{\lambda}_* > \lambda_*$ ($\boldsymbol{\lambda}_*$ is the critical parameter of another model) in order to prove the theorems. Condition (b) was proved considering as observable only cylindrical functions (strictly local observables).

We prove that (a) and (b) hold for the whole supercritical region $\lambda > \lambda_*$, even when one observes simultaneously translations of a cylindrical function. As a corollary, condition (b) holds also if one observes averages of the translations of a cylindrical function, like the spatial density of particles (macroscopic observables).

In Section 2 we define the contact process and related processes, and summarize also some properties of those processes which we use later. In Section 3 we state the main theorems which are proven in Sections 4 and 5. In Section 6 we prove that condition (a) does not hold for $\lambda < \lambda_*$.

2. THE CONTACT MODEL

The contact model of Harris[2-4] was motivated by a biological problem: the propagation of an infection. We deal with it because it has some important features: spatial structure (nearest-neighbor interaction), and even in one dimension presents critical behavior. In fact this model is also known in the physics literature[7] with a different name. In this context it is studied numerically in connection with reggeon field theory.

The model is a continuous time Markov process taking its values on the set $\mathscr{P}(\mathbb{Z})$ of all the subsets of $\mathbb{Z}$ (we will restrict the definition to the one-dimensional case). Particles are distributed in $\mathbb{Z}$ in such a way that each site is empty or occupied by at most one particle. $\xi(t)$ denotes the set of occupied sites at time t. We construct the contact process with the help of a random graph (percolation structure), in the space-time diagram $\mathbb{Z} \times \mathbb{R}_+$. For each $i \in \mathbb{Z}$ consider three independent Poisson processes on $\mathbb{R}_+$: $(\vec{\tau}^i_n)_{n \in \mathbb{N}}$, $(\overleftarrow{\tau}^i_n)_{n \in \mathbb{N}}$, $(\tau^{+i}_n)_{n \in \mathbb{N}}$ with parameters λ, λ, and 1, respectively. We suppose that for i varying in $\mathbb{Z}$ the processes are all independent. Now, for each $i \in \mathbb{Z}$ we draw arrows in $\mathbb{Z} \times \mathbb{R}_+$, from $(i, \vec{\tau}^i_k)$ to $(i+1, \vec{\tau}^i_k)$, $k = 1, 2,..., i \in \mathbb{Z}$. Secondly we draw arrows from $(i, \overleftarrow{\tau}^i_k)$ to $(i-1, \overleftarrow{\tau}^i_k)$, $k = 1, 2,..., i \in \mathbb{Z}$. Finally we put down $+$ signs at each of the points (i, τ^{+i}_k), $k = 1, 2,..., i \in \mathbb{Z}$.

We call a segment linking (x, t) to (x, s) a time segment. We give it the orientation from (x, t) to (x, s) if $s > t$.

Given two points (i, s) and (j, t) in the space-time $\mathbb{Z} \times \mathbb{R}_+$, with $s < t$, we say that there is a path from (i, s) to (j, t) if there is a connected chain of oriented time segments and arrows in the random graph, leading from (i, s) to (j, t), following the direction of the time segments and arrows and without passing through a $+$ sign.

Now, given $A \in \mathscr{P}(\mathbb{Z})$, we define the process $[\xi^A(t), t \geq 0]$ in the following way: $\xi^A(0) = A$, and for $t > 0$, $\xi^A(t) = \{j \in \mathbb{Z}$: there is a path from $(i, 0)$ to (j, t), for some $i \in A\}$.

Using the same percolation structure we define other related processes. The contact process taking values on $\mathscr{P}(\{-N, -N+1,...\})$ or on $\mathscr{P}(\{-N, -N+1,..., N-1, N\})$, where N is a positive integer. In the first case it is enough to use just the Poisson processes $(\vec{\tau}^i_n)_{n \in \{-N, -N+1,...\}}$, $(\overleftarrow{\tau}^i_n)_{n \in \{-N+1,...\}}$ and $(\tau^+_n)_{n \in \{-N, -N+1,...\}}$ disregarding the others. In this case we say that there is a path from (i, s) to (j, t), $t > s$ if it can be constructed only with the arrows determined by this processes. In the same way as before, but with this new definition of path, we define for $t > 0$, $\xi^A_{[-N,\infty)}(t) = \{j \in \mathbb{Z}$: there is a path form $(i, 0)$ to (j, t) for some $i \in A\}$, $\xi^A_{[-N,\infty)}(0) = A$, where $A \subset \{-N, -N+1,...\}$.

Analogously we define the contact process on $\mathscr{P}(\{-N, -N+1,..., N-1, N\})$ using to construct the paths only the Poisson processes $(\vec{\tau}^i_n)_{n \in \{-N,...,N-1\}}$, $(\overleftarrow{\tau}^i_n)_{n \in \{-N+1,...,N\}}$ and $(\tau^{+i}_n)_{n \in \{-N,...,N\}}$. We will use for it the notation $(\xi^A_N(t), t \geq 0)$, for any $A \subset \{-N,..., N\}$ as initial state.

In this way we have constructed all those processes on the same probability space, and we have some useful relations, such as

$$\xi_N^A(t) \subset \xi^A(t) \ \forall A \subset \{-N,..., N\}, \qquad t>0$$
$$\xi_{[-N,\infty)}^A(t) \subset \xi^A(t) \ \forall A \subset \{-N,...\}, \qquad t>0$$
$$\xi^A(t) \subset \xi^B(t) \qquad \text{if} \quad A \subset B \subset \mathbb{Z}$$

which hold for all possible trajectories of the processes. Relations like the last one are called monotonicity.[3] We use the convention of omitting the initial condition in the notation when it is the largest possible. So $\xi(t)=\xi^{\mathbb{Z}}(t)$, $\xi_N(t)=\xi_N^{\{-N,...,N\}}(t)$, $\xi_{[-N,\infty)}(t)=\xi_{[-N,\infty)}^{\{-N,...\}}(t)$. Expectations will be denoted by $E(\cdot)$.

By elementary Markov process theory and the fact that $\varnothing$ is a trap $(\xi_N^A(t), t\geqslant 0)$ is ergodic with invariant measure concentrated at the empty set $\delta_\varnothing$. For $(\xi^A(t), t\geqslant 0)$ and $(\xi_{[-N,\infty)}^A(t), t\geqslant 0)$ the situation is different. There is $\lambda_* \in \mathbb{R}_+$ such that if $\lambda<\lambda_*$ both are ergodic, if $\lambda>\lambda_*$ both are not ergodic. In the second case there are for both just two extremal invariant measures: one is $\delta_\varnothing$ and the other, which we denote, respectively, by μ and $\mu_{[0,\infty)}$, can be obtained as the weak limits $\xi(t)\to\mu$, $\xi_{[-N,\infty)}(t)\to\mu_{[-N,\infty)}$ as $t\to\infty$. The fact that the critical value λ_* is the same for both models is stated in Ref. 2. It is not proved there but the proof is simple using the techniques developed there.

Now we summarize other properties of the contact process which we use later. For more details and the proofs see Refs. 2, 3, 4, and 8. Analogous statements hold for the semi-infinite process $(\xi_{[0,\infty)}(t), t\geqslant 0)$.

(1) Monotone convergence: If $s>r$ then for any $A\subset\mathbb{Z}$ finite, $P(\xi(r)\cap A\neq\varnothing)\geqslant P(\xi(s)\cap A\neq\varnothing)\geqslant \mu(\eta\colon \eta\cap A\neq\varnothing)$.

(2) Self-duality: If A, $B\subset\mathbb{Z}$ and at least one of them is finite, then for any $t\geqslant 0$

$$P(\xi^A(t)\cap B\neq\varnothing)=P(\xi^B(t)\cap A\neq\varnothing)$$

In particular for finite $A\subset\mathbb{Z}$

$$P(\xi^A(t)\neq\varnothing)=P(\xi(t)\cap A\neq\varnothing)$$

A useful consequence of self-duality if $\lambda>\lambda_*$ is the following: Since $\varnothing$ is a trap and $\xi(t)\to\mu$ weakly as $t\to\infty$

$$P(\xi^A(s)\neq\varnothing, \forall s\geqslant 0)=\lim_{t\to\infty} P(\xi^A(t)\neq\varnothing)$$
$$=\lim_{t\to\infty} P(\xi(t)\cap A\neq\varnothing)=\mu(\eta\colon \eta\cap A\neq\varnothing)$$

for any finite $A\subset\mathbb{Z}$.

For $\lambda < \lambda_*$, by the same reasoning one concludes that for any finite $A \subset \mathbb{Z}$

$$P(\xi^A(s) \neq \varnothing, \forall s \geqslant 0) = 0$$

(3) Define $\rho_\lambda = P(\xi^{\{0\}}(s) \neq \varnothing, \forall s \geqslant 0)$. Then if $\lambda < \lambda_*$, $\rho_\lambda = 0$ and if $\lambda > \lambda_*$, $\rho_\lambda = \mu(\eta: \eta \cap \{0\} \neq \varnothing) > 0$.

(4) μ is translation invariant and ergodic under space translations. In fact it has even exponentially decaying correlations.

3. RESULTS

We show that for $\lambda > \lambda_*$ the process $(\xi_N(t), t \geqslant 0)$ behaves metastably for large N, in the sense of conditions (a) and (b) of the Introduction. Informally its behavior is as follows: Initially there is a global phenomenon in which $\xi_N(t)$ becomes close to μ restricted to $\mathscr{P}(\{-N,..., N\})$. As $\lambda > \lambda_*$, the tendency is of expansion, i.e., if we were considering the process $(\xi^{\{-N,...,N\}}(t), t \geqslant 0)$, with large N, with great probability $\min \xi^{\{-N,...,N\}}(t) \to -\infty$ and $\max \xi^{\{-N,...,N\}}(t) \to +\infty$ as $t \to \infty$. Nevertheless the boundary conditions at N and $-N$ prevent this expansion. The system remains in apparent equilibrium with $\xi_N(t)$ close to μ restricted to $\mathscr{P}(\{-N,..., N\})$ until a great fluctuation carries it to the true equilibrium at $\delta_\varnothing$.

Let us introduce some notation:

$$T_N^A = \inf\{t > 0: \xi_N^A(t) = \varnothing\}$$

$$T_N = \inf\{t > 0: \xi_N(t) = \varnothing\}$$

As usually we identify $\mathscr{P}(\mathbb{Z})$ with $\{0, 1\}^{\mathbb{Z}}$, if $\eta \in \mathscr{P}(\mathbb{Z})$ we write for any $x \in \mathbb{Z}$, $\eta(x) = 1$ if $x \in \eta$ and $\eta(x) = 0$ if $x \notin \eta$. We define $\max \eta = \sup\{x \in \mathbb{Z}: \eta(x) = 1\}$, $\min \eta = \inf\{x \in \mathbb{Z}: \eta(x) = 1\}$. Given a cylindrical function $f: \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$, the support of f (defined as the smallest $B \subset \mathbb{Z}$ such that $f(A) = f(A \cap B)$, $\forall A \subset \mathbb{Z}$) will be denoted by $\Lambda(f)$ or just by Λ if there is no possibility of confusion. We define the operators of translation τ_i, $i \in \mathbb{Z}$ by

$$(\tau_i f)(\eta) = f(\eta^{(i)}), \qquad \eta^{(i)}(x) = \eta(x - i)$$

Given f and two numbers N, $L \in \mathbb{N}$, we define

$$I_{f,N}(L) = \{i \in \mathbb{Z}: \Lambda(\tau_i f) \subset [-N + L, N - L] \cap \mathbb{Z}\}$$

$$\tilde{I}_{f,N} = \{i \in \mathbb{Z}: \Lambda(\tau_i f) \subset [-N, N] \cap \mathbb{Z}\}$$

Temporal means of f with respect to the process $(\xi_N(t), t \geqslant 0)$ will be represented by

$$A_R^N(s, f) = R^{-1} \int_s^{s+R} f(\xi_N(t))\, dt$$

where s is the instant of the beginning of the measurement and R the time interval of observation. Taking the spatial mean of translations of f we define

$$\bar{f} = \frac{1}{|\tilde{I}_{f,N}|} \sum_{i \in \tilde{I}_{f,N}} \tau_i(f)$$

$$B_R^N(s, f) = A_R^N(s, \bar{f}) = \frac{1}{|\tilde{I}_{f,N}|} \sum_{i \in \tilde{I}_{f,N}} A_R^N(s, \tau_i f)$$

Given a probability measure ν on $\mathscr{P}(\mathbb{Z})$ we write $\nu(f) = \int f\, d\nu$ for the expectation of f with respect to ν. By the definition of $(\xi_N(t), t \geqslant 0)$, $P(T_N > t)$ is a continuous and strictly decreasing function of t. So there exists a unique β_N such that $P(T_N > \beta_N) = e^{-1}$.

It is clear that as N increases, $P(T_N \leqslant t) \to 0$ for any $t > 0$. In order to see the jump to the stable situation one has to consider a different time scale. Let us refer to the original time scale with which we have been dealing up to now as the *microscopic time scale*. We introduce now the macroscopic time = (microscopic time)/β_N.

Theorem 1 below states that condition (a) in the introduction holds (asymptotically as $N \to \infty$) for the macroscopic time. At the end of Section 4 we show that it is also possible to use the expected value $E(T_N)$ instead of β_N to rescale time. Now, the meaning of condition (b) in the Introduction is the existence of a time scale intermediate between the microscopic and macroscopic one: intermediate time = (microscopic time)/R_N,

$$1 \ll R_N \ll \beta_N$$

such that for any cylindrical f

$$A_{R_N}^N(s, f) \cong \mu(f)$$

if

$$s + R_N < T_N$$

More precisely we state the following:

Theorem 1. If $\lambda > \lambda_*$ then T_N/β_N converges in distribution to a unit mean exponential random variable, as $N \to \infty$.

Theorem 2. If $\lambda > \lambda_*$, there is a sequence of positive real numbers $(R_N, N \geqslant 1)$ such that (i) $R_N/\beta_N \to 0$ as $N \to \infty$ and (ii) for all $\varepsilon > 0$ and $f: \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$ cylindrical, there is $L = L(\varepsilon, f) \in \mathbb{N}$, independent of N, such that

$$P[\max_{\substack{l \in \mathbb{Z} \\ 0 \leqslant l < K_N}} \max_{i \in I_{f,N}(L)} |A^N_{R_N}(lR_N, \tau_i f) - \mu(f)| > \varepsilon] \to 0$$

as $N \to \infty$, where $K_N = \max\{l \in \mathbb{Z}^+ : lR_N < T_N\}$.

Theorem 3. If $\lambda > \lambda_*$ and $(R_N, N \geqslant 1)$ is a sequence satisfying the conditions of Theorem 2 above then

$$P[\max_{\substack{l \in \mathbb{Z} \\ 0 \leqslant l < K_N}} |B^N_{R_N}(lR_N, f) - \mu(f)| > 0] \to 0$$

as $N \to \infty$.

An important particular case of Theorem 3 is that with $f = I_{\{\eta: \eta(0) = 1\}}$, then $\bar{f}$ is the spatial density of particles. The proof of Theorem 2 depends on estimating the decay of temporal correlations of $(\xi(t), t \geqslant 0)$.

4. PROOF OF THEOREM 1

We will show that for all $s > 0$, $t > 0$

$$\lim_{N \to \infty} \left| P\left[\frac{T_N}{\beta_N} > s + t\right] - P\left[\frac{T_N}{\beta_N} > s\right] P\left[\frac{T_N}{\beta_N} > t\right] \right| = 0 \tag{4.1}$$

This implies the result since by induction it implies that for positive rational r

$$P\left(\frac{T_N}{\beta_N} > r\right) \to e^{-r}$$

and by monotonicity the same follows for all positive real values of r.

In order to prove (4.1) we define for $b > 0$

$$F_b = \left\{A \subset \mathbb{Z}: \frac{|A \cap [-b, 1]|}{b} > \frac{\rho}{2}, \frac{|A \cap [1, b]|}{b} > \frac{\rho}{2}\right\} \tag{4.2}$$

where $\rho = \rho_\lambda = \mu\{\eta: 0 \in \eta\}$, $\rho > 0$ if $\lambda > \lambda_*$.

By monotonicity it is clear that

$$P\left[\frac{T_N^A}{\beta_N}>x\right]\leqslant P\left[\frac{T_N}{\beta_N}>x\right]$$

for any $A\subset\{-N,..., N\}$ and $x\in\mathbb{R}$.

So

$$P\left[\frac{T_N}{\beta_N}>s\right]P\left[\frac{T_N}{\beta_N}>t\right]\geqslant P\left[\frac{T_N}{\beta_N}>s+t\right] \tag{4.3}$$

But

$$\begin{aligned}
&P\left[\frac{T_N}{\beta_N}>s+t\right]\\
&\quad=\sum_{\substack{A\subset\{-N,...,N\}\\ A\neq\varnothing}} P\left[\frac{T_N}{\beta_N}>s+t \,\middle|\, \xi_N(\beta_N s)=A\right]\cdot P[\xi_N(\beta_N s)=A]\\
&\quad=\sum_{\substack{A\subset\{-N,...,N\}\\ A\in F_b}} P\left[\frac{T_N^A}{\beta_N}>t\right]\cdot P[\xi_N(\beta_N s)=A]\\
&\qquad+\sum_{\substack{A\subset\{-N,...,N\}\\ A\notin F_b\\ A\neq\varnothing}} P\left[\frac{T_N^A}{\beta_N}>t\right]\cdot P[\xi_N(\beta_N s)=A]
\end{aligned}$$

and using (4.3)

$$\begin{aligned}
&\left|P\left[\frac{T_N}{\beta_N}>s\right]P\left[\frac{T_N}{\beta_N}>t\right]-P\left[\frac{T_N}{\beta_N}>s+t\right]\right|\\
&\quad\leqslant P\left[\frac{T_N}{\beta_N}>s\right]P\left[\frac{T_N}{\beta_N}>t\right]-\sum_{\substack{A\subset\{-N,...,N\}\\ A\in F_b}} P\left[\frac{T_N^A}{\beta_N}>t\right]P[\xi_N(\beta_N s)=A]\\
&\quad\leqslant P\left[\frac{T_N}{\beta_N}>s\right]P\left[\frac{T_N}{\beta_N}>t\right]-\min_{\substack{A\subset\{-N,...,N\}\\ A\in F_b}}\left(P\left[\frac{T_N^A}{\beta_N}>t\right]\right)P[\xi_N(\beta_N s)\in F_b]\\
&\quad=P\left[\frac{T_N}{\beta_N}>s\right]\left\{P\left[\frac{T_N}{\beta_N}>t\right]-\min_{\substack{A\subset\{-N,...,N\}\\ A\in F_b}} P\left[\frac{T_N^A}{\beta_N}>t\right]\right\}\\
&\qquad+\min_{\substack{A\subset\{-N,...,N\}\\ A\in F_b}} P\left[\frac{T_N^A}{\beta_N}>t\right]\left\{P\left[\frac{T_N}{\beta_N}>s\right]-P[\xi(\beta_N s)\in F_b]\right\}
\end{aligned}$$

$$\leqslant \left\{ P\left[\frac{T_N}{\beta_N} > t\right] - \min_{\substack{A \subset \{-N,\dots,N\} \\ A \in F_b}} P\left[\frac{T_N^A}{\beta_N} > t\right]\right\}$$

$$+ P\left[\frac{T_N}{\beta_N} > s,\ \xi(\beta_N s) \notin F_b\right]$$

The relation (4.1) will be proven once we show that for all $\varepsilon > 0$, there exist $b(\varepsilon)$ and $N(\varepsilon) > b(\varepsilon)$ such that

$$P\left[\frac{T_N}{\beta_N} > t\right] - P\left[\frac{T_N^A}{\beta_N} > t\right] < \varepsilon \qquad \text{if} \quad N \geqslant N(\varepsilon) \text{ and } A \in F_{b(\varepsilon)} \tag{4.4}$$

$$P\left[\frac{T_N}{\beta_N} > s,\ \xi(\beta_N s) \notin F_{b(\varepsilon)}\right] < \varepsilon \qquad \text{if} \quad N \geqslant N(\varepsilon) \tag{4.5}$$

To prove (4.4) we consider ξ_N and ξ_N^A constructed with the same percolation structure. Then

$$P\left[\frac{T_N}{\beta_N} > t\right] - P\left[\frac{T_N^A}{\beta_N} > t\right] = P\left[\frac{T_N}{\beta_N} > t,\ \frac{T_N^A}{\beta_N} \leqslant t\right] \leqslant P[T_N \neq T_N^A] \tag{4.6}$$

For $\lambda > \lambda_*$, given $\varepsilon > 0$ there is $n(\varepsilon)$ such that if $n \geqslant n(\varepsilon)$ then

$$\mu_{[0,\infty)}\{B\colon B \cap [1, n] = \varnothing\} \leqslant \frac{\varepsilon}{2}$$

If N is large enough we can take $b = b'(\varepsilon)$ such that $n(\varepsilon) \leqslant b \cdot \rho/2$, $b < N$. It follows for $A \in F_b$ that $|A \cap [-b, -1]| \geqslant b \cdot \rho/2 \geqslant n(\varepsilon)$. So

$$P[T_{[-N,\infty)}^{A \cap [-b,-1]} = \infty] \geqslant P[T_{[-N,\infty)}^{\{-N,\dots,-N+n(\varepsilon)\}} = \infty] \geqslant 1 - \frac{\varepsilon}{2} \tag{4.7a}$$

where the first inequality is proved in the same way as relation (16) in Ref. 2 and the second is a consequence of the self-duality of the contact processes.[3,4]

Analogously,

$$P[T_{[-\infty,N]}^{A \cap [1,b]} = \infty] \geqslant 1 - \frac{\varepsilon}{2} \tag{4.7b}$$

We define the event

$$E = [T_{[-N,\infty)}^{A \cap [-b,-1]} = T_{(-\infty,N]}^{A \cap [1,b]} = \infty]$$

and the stopping times

$$U = \inf\{t > 0 \colon N \in \xi_{[-N,\infty)}^{A \cap [-b,-1]}(t)\}$$
$$V = \inf\{t > 0 \colon -N \in \xi_{(-\infty,N]}^{A \cap [1,b]}(t)\}$$

On E, $T_N > T_N^A > \max(U, V)$. By standard arguments for the contact processes, based on the fact that the interaction is between nearest neighbors, we have that for $t > \max(U, V)$

$$\xi_N(t) = \xi_N^A(t)$$

So, on E, $T_N = T_N^A$ and (4.6) and (4.7) imply (4.4).

To prove (4.5) we construct $(\xi_N(t), t \geqslant 0)$ and $(\xi(t), t \geqslant 0)$ using the same percolation structure. We take b and L such that $b < N - L < N$, then

$$\begin{aligned} &P[\xi_N(\beta_N s) \notin F_b, T_N > \beta_N s] \\ &\quad \leqslant P[\xi_N(\beta_N s) \notin F_b, T_N > \beta_N s, \min \xi_N(\beta_N s) < -N + L, \\ &\qquad \max \xi_N(\beta_N s) > N - L] \\ &\qquad + P[\min \xi_N(\beta_N s) \geqslant -N + L, T_N > \beta_N s] \\ &\qquad + P[\max \xi_N(\beta_N s) \leqslant N - L, T_N > \beta_N s] \end{aligned} \tag{4.8}$$

But on $[T_N > \beta_N s]$

$$\begin{aligned} &\xi_N(\beta_N s) \cap [\min \xi_N(\beta_N s), \max \xi_N(\beta_N s)] \\ &\quad = \xi(\beta_N s) \cap [\min \xi_N(\beta_N s), \max \xi_N(\beta_N s)] \end{aligned} \tag{4.9}$$

Thus the first summand in (4.8) may be bounded above by $P[\xi(\beta_N s) \notin F_b]$. But by the ergodicity of μ, there is $b''(\varepsilon)$ such that if $b > b''(\varepsilon)$

$$P(\xi(\beta_N s) \notin F_b) \leqslant \mu(F_b^c) \leqslant \frac{\varepsilon}{3} \tag{4.10}$$

where we used the stochastic monotonicity of the convergence $\xi(t) \to \mu$, as $t \to \infty$.

Finally, we will control the other two terms in (4.8) using again the semi-infinite contact process. From the nearest-neighbor nature of the interaction if follows that on $[T_N > t]$,

$$\min \xi_N(t) = \min \xi_{[-N,\infty)}(t)$$

So

$$P[\min \xi_N(\beta_N s) \geqslant -N+L,\ T_N > \beta_N s]$$
$$\leqslant P[\min \xi_{[-N,\infty)}(\beta_N s) \geqslant -N+L]$$
$$\leqslant \mu_{[-N,\infty)}\{A \subset [-N, \infty) \cap Z: A \cap [-N, -N+L-1] = \varnothing\} \quad (4.11)$$

where the last inequality followed from the monotone convergence of the law of $\xi_{[-N,\infty)}(t)$ to $\mu_{[-N,\infty)}$ as $t \to \infty$. As $\lambda > \lambda_*$, we can take $L(\varepsilon)$ such that for $L > L(\varepsilon)$, the right-hand side of (4.11) is smaller than $\varepsilon/3$.

The other summand in (4.8) is analogous and (4.4) and (4.5) are proved.

Remark. We can use $E(T_N)$ instead of β_N to rescale the time. This follows from the following proposition.

Proposition 1. $E(T_N)/\beta_N \to 1$ as $N \to \infty$.

Proof.

$$\frac{E(T_N)}{\beta_N} = \frac{1}{\beta_N} \int_0^\infty P(T_N > t)\, dt = \int_0^\infty P\left(\frac{T_N}{\beta_N} > t\right) dt$$

but $P(T_N/\beta_N > t) \leqslant P(T_N/\beta_N > [t]) \leqslant e^{-[t]}$, where $[t]$ is the integer part of t. By the dominated convergence theorem it follows that

$$\lim_{N\to\infty} \frac{E(T_N)}{\beta_N} = \int_0^\infty \lim_{N\to\infty} P\left(\frac{T_N}{\beta_N} > t\right) dt = \int_0^\infty e^{-t}\, dt = 1 \quad \blacksquare$$

5. THERMALIZATION

First we prove that the time correlations of $(\xi(t), t \geqslant 0)$ decay exponentially fast.

Theorem 4. For any cylindrical function $f: \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$ there are constants $C > 0$, $\gamma > 0$, such that

$$|\text{cov}[f(\xi(r)), f(\xi(s))]| \leqslant Ce^{-\gamma|s-r|}$$

Proof. Without loss of generality we consider $s > r$. We use the notation

$$u = |s-r|$$

given $A \subset \mathbb{Z}$, $\Delta(A) = \{\eta \in \mathscr{P}(A): \eta \cap A \neq \varnothing\}$, $I_{\Delta(A)}(\cdot) = \text{indicator of } \bar{A}$.

As any cylindrical function is a finite linear combination of these indicators it is enough to prove for any pair $A, B \subset \mathbb{Z}$, $|A| < \infty$, $|B| < \infty$, that

$$|\text{cov}[I_{\Delta(A)}(\xi(r)), I_{\Delta(B)}(\xi(s))]| \leqslant Ce^{-\gamma u}$$

We will construct some auxiliary processes. First we define dual percolation structure (for more details about this technique see Ref. 3). Consider the percolation structure (random graph) constructed in Section 2. We define an inverted (microscopic) time scale $l = s - t$. Invert the direction of the time segments so that they are oriented according to increasing l. Invert also the direction of the arrows. Using l as time scale and given two points (i, l_1), $(j, l_2) \in \mathbb{Z} \times (-\infty, s]$ such that $l_1 < l_2$, we say that there is an inverted path from (i, l_1) to (j, l_2) if there is a connected chain of time segments and arrows leading from (i, l_1) to (j, l_2), following the new orientations of the time segments and arrows.

Now consider the processes $(X_l, 0 \leqslant l \leqslant s)$ and $(Y_l, u \leqslant l \leqslant s)$ defined by (we are using l as time scale) the following:

$$X_l = \{j \in \mathbb{Z}\text{: there is an inverted path from } (i, 0) \text{ to } (j, l) \text{ for some } i \in B\}$$

$$Y_l = \{j \in \mathbb{Z}\text{: there is an inverted path from } (i, u) \text{ to } (j, l) \text{ for some } j \in A\}$$

The processes $(X_l, 0 \leqslant l \leqslant s)$ and $(Y_l, u \leqslant l \leqslant s)$ have, respectively, the same laws of $(\xi^B(t), 0 \leqslant t \leqslant s)$ and $(\xi^A(t), 0 \leqslant t \leqslant r)$, the first under the correspondence $l \to t$ and the second under $l \to t + u$.

We define the events

$$E_A = [I_{\Delta(A)}(\xi(r)) = 1] = [Y_s \neq \varnothing]$$

$$E_B = [I_{\Delta(B)}(\xi(s)) = 1] = [X_s \neq \varnothing]$$

$$F = [X_u \neq \varnothing]$$

Now, by the independence in disjoint sets property of the Poisson processes, the events E_A and F are independent. Then

$$\begin{aligned}
&|\operatorname{cov}[I_{\Delta(A)}(\xi(r)), I_{\Delta(B)}(\xi(s))]| \\
&\quad = |P(E_A \cap E_B) - P(E_A)\, P(E_B)| \\
&\quad = |P(E_A \cap E_B \cap F) - P(E_A)\, P(E_B \cap F)| \\
&\quad = |P(E_A \cap E_B \cap F) - P(E_A \cap F) + P(E_A)\, P(F) - P(E_A)\, P(E_B \cap F)| \\
&\quad = |P(E_A)\, P(E_B^C \cap F) - P(E_A \cap F \cap E_B^C)| \leqslant P(E_B^C \cap F) \\
&\quad = P(X_u \neq \varnothing, X_s = \varnothing) = P(u < T^B < s) \\
&\quad \leqslant P(u < T^B < \infty) \leqslant C e^{-\gamma u}
\end{aligned}$$

where the last inequality was proved in Ref. 2 (Theorem 5). ∎

Remark. In the same way one can prove analogous statements for the process $(\xi_{[0,\infty)}(t), t \geqslant 0)$:

$$|\mathrm{cov}[f(\xi_{[0,\infty)}(r)), f(\xi_{[0,\infty)}(s))]| \leqslant Ce^{-\gamma|s-r|}$$

Proof of Theorem 2. Since T_N is almost surely finite, for any positive number R_N, K_N (as defined in the statement of the theorem) is a well-defined and finite random variable with values in $\mathbb{N}$. Moreover, if the R_N verify condition (i) above it follows from Theorem 1 that $P(T_N < R_N) \to 0$ as $N \to \infty$, i.e., $P(K_N = 0) \to 0$. Let us now assume R_N is a sequence satisfying (i). For $\varepsilon > 0$ and f cylindrical given, let

$$B_{k,i}^N = [|A_{R_N}^N(kR_N, \tau_i f) - \mu(f)| > \varepsilon]$$

Then, for any integers $m \geqslant 1$, $L \geqslant 0$

$$\begin{aligned}
P&\left[K_N \geqslant 1, \bigcap_{0 \leqslant k < K_N} \bigcap_{i \in I_{f,N}(L)} (B_{k,i}^N)^C\right] \\
&= \sum_{j=1}^{\infty} \left(P[K_N = j] - P\left[\bigcup_{k=0}^{j-1} \bigcup_{i \in I_{f,N}(L)} B_{k,i}^N, K_N = j\right]\right) \\
&\geqslant P[1 \leqslant K_N \leqslant m] - \sum_{j=1}^{m} P\left[\bigcup_{k=0}^{j-1} \bigcup_{i \in I_{f,N}(L)} B_{k,i}^N, K_N = j\right] \\
&\geqslant P[1 \leqslant K_N \leqslant m] - \sum_{j=1}^{m} j(2N+1) \max_{\substack{0 < k < j \\ i \in I_{f,N}(L)}} P[B_{k,i}^N, K_N = j] \\
&\geqslant P[1 \leqslant K_N \leqslant m] - m^2(2N+1) \max_{j \geqslant 1} \max_{\substack{0 \leqslant k < j \\ i \in I_{f,N}(L)}} P[B_{k,i}^N, K_N = j]
\end{aligned} \tag{5.1}$$

We construct $(\xi_N(t), t \geqslant 0)$, $(\xi(t), t \geqslant 0)$, $(\xi_{[-N,\infty)}(t), t \geqslant 0)$, and $(\xi_{(-\infty,N]}(t), t \geqslant 0)$ using the same percolation structure. Then we have the following relation between events if $0 < L < N$:

$$\begin{aligned}
&[\min \xi_N(t) < -N + L, \max \xi_N(t) > N - L] \\
&\qquad \subset \bigcap_{i \in I_{f,N}(L)} [\tau_i f(\xi_N(t)) = \tau_i f(\xi(t))]
\end{aligned} \tag{5.2}$$

$$[T_N > t] \subset [h_L(\xi_N(t)) = h_L(\xi_{[-N,\infty)}(t))] \tag{5.3}$$

where $h_L(\eta) = I_{\{\xi: \xi \cap [-N, -N+L] = \varnothing\}}(\eta)$.

In the case $k<j$ and $i\in I_{f,N}(L)$

$$\begin{aligned}
&P[B^N_{k,i}, K_N=j]\\
&\quad =P[|A^N_{R_N}(kR_N,\tau_i f)-\mu(f)|>\varepsilon, K_N=j]\\
&\quad \leqslant P\left[\left|A^N_{R_N}(kR_N,\tau_i f)-\frac{1}{R_N}\int_{kR_N}^{(k+1)R_N}\tau_i f(\xi(t))\,dt\right|>\frac{\varepsilon}{2} \text{ or}\right.\\
&\quad\quad \left.\left|\frac{1}{R_N}\int_{kR_N}^{(k+1)R_N}\tau_i f(\xi(t))\,dt-\mu(f)\right|>\frac{\varepsilon}{2}, K_N=j\right]
\end{aligned}\tag{5.4}$$

But, by (5.2) and (5.3), if $k<j$ and $i\in I_{f,N}(L)$

$$\begin{aligned}
[K_N=j]\subset&\left[\left|A^N_{R_N}(kR_N,\tau_i f)-\frac{1}{R_N}\int_{kR_N}^{(k+1)R_N}\tau_i f(\xi(t))\,dt\right|\right.\\
&\leqslant 2\,\|f\|\,\frac{1}{R_N}\int_{kR_N}^{(k+1)R_N}[h_L(\xi_N(t))+h_L(S\xi_N(t))]\,dt\\
&\left.=2\,\|f\|\,\frac{1}{R_N}\int_{kR_N}^{(k+1)R_N}[h_L(\xi_{[-N,\infty)}(t))+h(S\xi_{(-\infty,N]}(t))]\,dt\right]
\end{aligned}\tag{5.5}$$

where $\|f\|=\sup_{\eta\subset\mathbb{Z}}f(\eta)$ and S is the operator defined by $(S\eta)(x)=\eta(-x)$.

We define now the events

$$\Gamma^R_{k,i}=\left[\left|\frac{1}{R}\int_{kR}^{(k+1)R}\tau_i f(\xi(t))\,dt-\mu(f)\right|>\frac{\varepsilon}{2}\right]$$

$$\tilde{\Gamma}^{R,L}_k=\left[2\,\|f\|\,\frac{1}{R}\int_{kR}^{(k+1)R}h_L(\xi_{[-N,\infty)}(t))\,dt>\frac{\varepsilon}{4}\right]$$

$$\Gamma^N_k=\Gamma^N_{k,0}$$

So (5.4) and (5.5) give us

$$\text{if } k<j \text{ and } i\in I_{f,N}(L) \text{ then}$$
$$P[B^N_{k,i}, K_N=j]\leqslant P(\Gamma^{R_N}_k)+2P(\tilde{\Gamma}^{R_N,L}_k)\tag{5.6}$$

Part (ii) of the theorem will be proved via (5.1) and (5.6) once part (i) is satisfied and we can find $L\in N$, and a sequence $(m_N, N\geqslant 1)$ such that as $N\to\infty$

(a) $P(1\leqslant K_N\leqslant m_N)\to 1$

(b) $m^2(2N+1)(\max_{k\geqslant 0}P(\Gamma^{R_N}_k)+\max_{k\geqslant 0}P(\tilde{\Gamma}^{R_N,L}_k))\to 0$

Condition (a) may be written as $P(T_N \leqslant m_N R_N) \to 1$ or, using Theorem 1, as $m_N R_N/\beta_N \to \infty$. Using the notation

$$\alpha_L(R) = \max_{k \geqslant 0} P(\Gamma_k^R) + \max_{k \geqslant 0} P(\tilde{\Gamma}_k^{R,L})$$

and including part (i) of the theorem, our problem now is to find $L \in \mathbb{N}$ and two sequences $(R_N, N \geqslant 1)$, $(m_N, N \geqslant 1)$ such that as $N \to \infty$

$$m_N R_N/\beta_N \to \infty \tag{5.7a}$$

$$m_N^2 N \alpha_L(R_N) \to 0 \tag{5.7b}$$

$$R_N/\beta_N \to 0 \tag{5.7c}$$

[the last relation is the condition (i) in the statement of the theorem]. Lemma 1 below shows that it is possible to choose $L = L(\varepsilon, f)$ such that $\alpha_L(R) \leqslant C/R$ (C depending on ε and f). Then it is enough to have (5.7a), (5.7c) and (remember that we have fixed $\varepsilon > 0$)

$$m_N^2 \frac{N}{R_N} \to 0 \tag{5.8}$$

Lemma 2 shows that $N/\beta_N \to 0$, then

$$m_N = (\beta_N/N)^{1/5} \tag{5.9a}$$

$$R_N = \beta_N^{9/10} N^{1/10} \tag{5.9b}$$

is a solutions to our problem. ∎

Lemma 1. If $\lambda > \lambda_*$ and $L = L(\varepsilon, f)$ is such that $\mu_{[0,\infty)}\{A: A \cap [0, L] = \varnothing\} \leqslant \varepsilon/16\, \|f\|$, $\exists R(\varepsilon, f)$ such that $\alpha_L(R) \leqslant C/R$ if $R > R(\varepsilon, f)$ (where C depends on ε and f).

Remark. We will use the convention that from expression to expression the value of C can change.

Proof. First we prove that

$$\max_{k \geqslant 1} P(\Gamma_k^N) \leqslant \frac{C}{R_N}$$

consider the random variables

$$X_k^R = \frac{1}{R} \int_{kR}^{(k+1)R} f(\xi(t))\, dt - \mu(f)$$

$$Y_k^R = \frac{1}{R} \int_{kR}^{(k+1)R} f(\xi(t))\, dt - \frac{1}{R} \int_{kR}^{(k+1)R} Ef(\xi(t))\, dt$$

Then $E(Y_k^R)=0$ and by Theorem 4

$$\operatorname{var}(Y_k^R)\leqslant\frac{2}{R^2}\int_{kR}^{(k+1)R}dr\int_{kR}^{(k+1)R}ds\,|\operatorname{cov}[f(\xi(r)),f(\xi(s))]|$$

$$\leqslant\frac{2}{R^2}\int_{kR}^{(k+1)R}dr\int_r^{\infty}ds\,Ce^{-\gamma(s-r)}=\frac{2}{R^2}\cdot CR=\frac{C}{R}$$

(C does not depend on k).

By the Chebyshev inequality, $\forall\delta>0$

$$P[|Y_k^N|>\delta]\leqslant\frac{C}{\delta^2 R}$$

On the other hand, $\xi(t)\to\mu$ weakly as $t\to\infty$, so $Ef(\xi(t))\to\mu(f)$. And

$$\frac{1}{R}\int_{kR}^{(k+1)R}Ef(\xi(t))\,dt\to\mu(f)$$

uniformly in k, as $R\to\infty$.

Given $\varepsilon>0$ we can take $R(\varepsilon)$ such that

$$R>R(\varepsilon,f)\Rightarrow\left|\frac{1}{R}\int_{kR}^{(k+1)R}Ef(\xi(t))\,dt-\mu(f)\right|\leqslant\frac{\varepsilon}{4}$$

for all $k\geqslant 1$.

For $R_N>R(\varepsilon,f)$ it follows that

$$P(\Gamma_k^R)=P\left[\,|X_k^R|>\frac{\varepsilon}{2}\right]\leqslant P\left[\,Y_k^R>\frac{\varepsilon}{4}\right]\leqslant\frac{C}{R_N}$$

The other term in $\alpha_L(R)$ can be controlled in an analogous way. We define

$$Z_k^R=\frac{1}{R}\int_{kR}^{(k+1)R}h_L(\xi_{[0,\infty)}(t))\,dt$$

$$W_k^R=Z_k^R-\frac{1}{R}\int_{kR}^{(k+1)R}Eh_L(\xi_{[0,\infty)}(t))\,dt$$

It folows that

$$E(W_k^R)=0$$

$$\operatorname{var}(W_k^R)\leqslant\frac{C}{R}$$

$$P[|W_k^R|>\delta]\leqslant\frac{C}{\delta^2 R}\qquad(\forall\delta>0)$$

Also $Eh_L(\xi_{[0,\infty)}(t)) \to \mu_{[0,\infty)}(h_L)$ as $t \to \infty$, increasing monotonically to the limit.

So, for any k and R

$$\frac{1}{R}\int_{kR}^{(k+1)R} Eh_L(\xi_{[0,\infty)}(t))\,dt \leqslant \mu_{[0,\infty)}(h_L) \leqslant \frac{\varepsilon}{16\,\|f\|}$$

Finally,

$$P(\tilde{\Gamma}_k^R) = P\left[2\,\|f\|\,|Z_k^R| > \frac{\varepsilon}{4}\right] \leqslant P\left[\,|W_k^R| > \frac{\varepsilon}{16\,\|f\|}\right] \leqslant \frac{C}{R} \quad \blacksquare$$

Lemma 2. $N/\beta_N \to 0$ as $N \to \infty$, if $\lambda > \lambda_*$.

Proof. We construct $(\xi_N(t), t \geqslant 0)$ and $(\xi(t), t \geqslant 0)$ on the same percolation structure. Consider the event

$$A_N = [\xi^{[0,N/2]\cap\mathbb{Z}}(t) \neq \varnothing, \forall t \geqslant 0]$$

By self-duality

$$P(A_N) = \mu\left\{B \subset \mathbb{Z}\colon B \cap \left[0, \frac{N}{2}\right] = \varnothing\right\} \to 0 \qquad \text{as} \quad N \to \infty$$

By standard arguments, on A_N the stopping time U_N defined by

$$U_N = \inf\{t > 0\colon N \in \xi^{(0,N/2]\cap\mathbb{Z}}(t)\}$$

is almost surely finite.

On A_N we have also the inequalities

$$Y_N \leqslant U_N \leqslant T_N$$

where Y_N is the instant of the $[N/2]$th occurrence of a Poisson process of rate λ.

So

$$\begin{aligned} P\left(T_N \leqslant \frac{\lambda}{2}\cdot\frac{N}{2}\right) &\leqslant P\left(T_N \leqslant \frac{\lambda}{2}\cdot\frac{N}{2}, A_N\right) + P(A_N^C) \\ &\leqslant P\left(Y_N \leqslant \frac{\lambda}{2}\cdot\frac{N}{2}, A_N\right) + P(A_N^C) \\ &\leqslant P\left(Y_N \leqslant \frac{\lambda N}{4}\right) + P(A_N^C) \to 0 \qquad \text{as} \quad N \to \infty \end{aligned} \tag{5.10}$$

where we used the law of large number for Y_N.

By Theorem 1,

$$P(T_N/\beta_N \leqslant x) \to (1-e^{-x})\, I_{[0,\infty)}(x)$$

This combined with (5.10) implies the thesis. ∎

Proof of Theorem 3. Theorem 2 implies

$$P\left[\max_{\substack{l \in \mathbb{Z} \\ 0 \leqslant l < K_N}} \left| \frac{1}{I_{f,N}(L)} \sum_{i \in I_{f,N}(L)} A^N_{R_N}(lR_N, f) - \mu(f) \right| > \frac{\varepsilon}{2} \right] \to 0$$

$$\text{as} \quad N \to \infty, \text{ if } L = L\left(\frac{\varepsilon}{2}, f\right) \tag{5.11}$$

Using the inequality

$$\left| \frac{a+b}{\alpha+\beta} - \frac{a}{\alpha} \right| = \left| \frac{\alpha b - \beta a}{\alpha(\alpha+\beta)} \right| \leqslant \left| \frac{b}{\alpha+\beta} \right| + \left| \frac{a}{\alpha} \frac{\beta}{\alpha+\beta} \right|$$

with $\alpha = |I_{A,N}(L)|$, $\beta = |\tilde{I}_{AN}| - |I_{A,N}(L)|$,

$$a = \sum_{i \in I_{f,N}(L)} A^N_{R_N}(lR_N, \tau_i f)$$

$$b = \sum_{i \in (\tilde{I}_{f,N} - I_{f,N}(L))} A^N_{R_N}(lR_N, \tau_i f)$$

we obtain

$$\left| \frac{1}{|\tilde{I}_{f,N}|} \sum_{i \in \tilde{I}_{f,N}} A^N_{R_N}(lR_N, \tau_i f) - \frac{1}{|I_{f,N}(L)|} \sum_{i \in I_{f,N}(L)} A^N_{R_N}(lR_N, \tau_i f) \right|$$

$$\leqslant \frac{4L\, \|f\|}{2N+1-|A|} \to 0 \qquad \text{as} \quad N \to \infty \tag{5.12}$$

The relations (5.11) and (5.12) imply the thesis. ∎

6. SUBCRITICAL CASE

We prove now that Theorem 1 is false in the subcritical case ($\lambda < \lambda_*$).

Theorem 6. If $\lambda < \lambda_*$, T_N/γ_N does not converge to an exponential random variable, for any sequence (γ_N, $N \geqslant 1$).

Proof. We show that $P(T_N/\ln N \leqslant x) \to 0$ if $x<1$ and that there exist $K>1$ such that $P(T_N/\ln N \leqslant x) \to 1$ if $x>K$.

The first part follows from the fact that $T_N \geqslant S_N = \max_{i=-N,\dots,N} \tau_1^{+i}$. So

$$\begin{aligned}
&P(T_N/\ln N \leqslant x)\\
&\quad \leqslant P(S_N/\ln N \leqslant x) = (1-e^{-x\ln N})^{2N+1}\\
&\quad = \left(1-\frac{1}{2N+1}\cdot\frac{2N+1}{N^x}\right)^{2N+1} \to 0 \qquad \text{as} \quad N\to\infty, \text{ if } x<1
\end{aligned}$$

To prove the second part we use Theorem 8 in Ref. 4. It states that there is $C=C(\lambda)$ such that

$$P(\max \xi_{(-\infty,0]}(t) > -e^{ct}) \to 0, \qquad \text{as} \quad N\to\infty$$

Now $\xi_N(t) \subset \xi_{(-\infty,N]}(t)$, so

$$\begin{aligned}
&P(T_N/\ln N > 2/C)\\
&\quad \leqslant P(\max \xi_N(2\ln N/C) > -N)\\
&\quad \leqslant P(\max \xi_{(-\infty,N]}(2\ln N/C) > -N)\\
&\quad = P(\max \xi_{(-\infty,0]}(2\ln N/C) > -2N) \to 0 \qquad \text{as} \quad N\to\infty \quad \blacksquare
\end{aligned}$$

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